Triangulations and a Generalization of Bose's Method

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Abstract

We present a nontrivial extension to Bose's method for the construction of Steiner triple systems, generalizing the traditional use of commutative and idempotent quasigroups to employ a new algebraic structure called a 3-tri algebra. Links between Steiner triple systems and 2-(v,3,3) designs via 3-tri algebras are also explored.

Keywords: Steiner triple system, quasigroup, latin square, Bose construction, Skolem construction, triangulation.

1 Background

Let X be a finite set. A *set system* or *configuration* is a pair $(X, \mathcal{A})$, where $\mathcal{A} \subseteq 2^X$. The *order* of the set system is $|X|$. The elements of X are *points* and the elements of $\mathcal{A}$ are *blocks*. A t-(v, k, λ) design is a k-uniform set system $(X, \mathcal{A})$ of order v such that every t-subset of X is contained in precisely λ blocks of $\mathcal{A}$. A 2-$(v, 3, 1)$ design is a *Steiner triple system* of order v and is denoted by STS(v). A (k, ℓ)-*configuration* in an STS $(X, \mathcal{A})$ is a subset of ℓ blocks in $\mathcal{A}$ whose union is a k-element subset of X. The *Pasch configuration* or *quadrilateral* is the (6,4)-configuration on elements (say) a, b, c, d, e, f with blocks $\{a, b, c\}$, $\{a, d, e\}$, $\{f, d, b\}$ and $\{f, c, e\}$. An STS is *anti-Pasch* (or *quadrilateral-free*) if it does not contain the (6,4)-configuration.

A *3-oriented graph* is a graph in which each edge e (with endpoints x and y) has one of three possible orientations: *positive*, *negative*, or *null oriented* from x to y. The edge e is positive oriented from x to y if and only if it is negative oriented from y to x; when e is null oriented the roles of x and y can be freely interchanged. We draw a positive oriented edge from x to y by an arrow from x to y and a null oriented edge without arrows. A 3-oriented graph is *simple* if, for every pair of vertices x and y, the graph contains at most one positive, one negative, and one null oriented edge from x to y. In a 3-oriented simple graph we can use without ambiguity $(x, y)^1$, $(x, y)^{-1}$, and $(x, y)^0$ to denote a positive, negative, and null oriented edge from x to y, respectively.

Let G be a 3-oriented simple graph. A path P in G through the vertices $x_0, \ldots, x_n$, $n \geq 1$, is denoted by $P = x_0, x_1^{\theta_1} \ldots, x_n^{\theta_n}$ where $\theta_1, \ldots, \theta_n \in \{1, -1, 0\}$, if and only if P uses the edges $(x_0, x_1)^{\theta_1}, \ldots, (x_{n-1}, x_n)^{\theta_n}$. When P is a cycle we write $P = (x_0^{\theta_0}, x_1^{\theta_1} \ldots, x_{n-1}^{\theta_{n-1}})$, with $\theta_0 = \theta_n$. If $\theta_0 + \theta_1 + \ldots + \theta_{n-1} \equiv 0 \bmod \lambda$ for some $\lambda > 0$, P is λ-*balanced.* A two-factor of G in which all cycles are λ-balanced is λ-*balanced.* A *triangulation* is a partition of the edges in G in paths of length 3, and a triangulation is *3-balanced* if all its paths are 3-balanced. As we soon see, 3-balanced triangulations of a 3-oriented simple graph are closely related to Steiner triple systems.

The graph with v vertices in which each pair of vertices is joined by three parallel edges is denoted by $3K_v$, and $3\overline{K}_v$ denotes the 3-oriented simple graph with v vertices in which each pair x and y of vertices is joined by a positive, a negative, and a null oriented edge from x to y. For both graphs, the vertex sets $V(3K_v) = V(3\overline{K}_v) = \{0, 1, \ldots, v-1\}$.

2 A generalization of Bose's method

Bose's method [1] is one of the most important and well known paradigms in design theory. Our objective is to develop a natural generalization.

Theorem 2.1 *Every 3-balanced triangulation of $3\overline{K}_v$ yields an STS($3v$).*

Proof: Let $\mathcal{T}$ be a 3-balanced triangulation of $3\overline{K}_v$. Let us define:

$$X = \{(a, i) | a \in \{0, \ldots, v-1\} \text{ and } i \in \{0, 1, 2\}\},$$

$$\mathcal{A}_1 = \{\{(a, 0), (a, 1), (a, 2)\} |\ a \in \{0, \ldots, v-1\}\}$$

and for each $T = (a^{\theta_a}, b^{\theta_b}, c^{\theta_c}) \in \mathcal{T}$

$$\mathcal{A}_T = \{\{(a, j), (b, (j + \theta_b) \bmod 3), (c, (j + \theta_b + \theta_c) \bmod 3)\} | j = 0, 1, 2\}.$$

$\mathcal{A}_T$ is well-defined, since if we use a different representation of T, say $(b^{\theta_b}, c^{\theta_c}, a^{\theta_a})$, we get:

$$A'_T = \{\{(b, k), (c, (k + \theta_c) \bmod 3), (a, (k + \theta_c + \theta_a) \bmod 3)\} | k = 0, 1, 2\},$$

Making the change of variable $k = (j + \theta_b) \bmod 3$, and applying the fact that $\theta_a + \theta_b + \theta_c \equiv 0 \bmod 3$, we find that $A'_T = A_T$. The other representations of T produce the same set.

We claim that $(X, \mathcal{A})$ with $\mathcal{A} = \mathcal{A}_1 \bigcup (\bigcup_{T \in \mathcal{T}} \mathcal{A}_T)$ is an STS($3v$). In fact, let $B = \{(a, i), (b, j)\}$ be a two-subset of X; if $a = b$ then $\{(a, 0), (a, 1), (a, 2)\}$ is the unique block in $\mathcal{A}$ containing B; otherwise B is contained in exactly one of the blocks in $\mathcal{A}_T$ where T is the unique triangle in $\mathcal{T}$ containing the edge $(a, b)^{(j-i) \bmod 3}$. ■

Bose's method builds Steiner triple systems using a special type of 3-balanced triangulations of $3\overline{K}_v$. A *Bose triangulation* is a 3-balanced triangulation of $3\overline{K}_v$ such that each of its triangles can be expressed as (a^0, b^1, c^{-1}) for appropriate elements $a, b, c \in \{0, \ldots, v-1\}$.

A *latin square* of order n is an $n \times n$ array, each cell of which contains exactly one of the symbols in $\{0, \ldots, n-1\}$, such that each row and each column of the array contains the symbols in $\{0, \ldots, n-1\}$ exactly once. A *quasigroup* of order n is a pair $(Q, \circ)$, where Q is a set of size n and $\circ$ is a binary operation on Q such that for every pair of elements $a, b \in Q$, the equations $a \circ x = b$ and $y \circ a = b$ have unique solutions. The tabular representation of a quasigroup of order n is a latin square of order n.

Proposition 2.2 *Every Bose triangulation produces a commutative and idempotent quasigroup. Conversely every commutative and idempotent quasigroup produces a Bose triangulation.*

Proof Let $\mathcal{T}$ be a Bose triangulation of $3\overline{K}_v$. If $Q = \{0, \ldots, v-1\}$ and $a, b \in Q$ we define

$$a \circ b = \begin{cases} c & \text{if } (a^0, c^1, b^{-1}) \in \mathcal{T} \\ a & \text{if } a = b \end{cases}$$

The binary operation $\circ$ is defined for every pair $a, b \in Q$ because there exists exactly one triangle in $\mathcal{T}$ containing the edge $(a, b)^0$. The operation $\circ$ is commutative and idempotent, as follows. The equation $a \circ x = b$ has only one solution in x because only the triangle (a^0, b^1, x^{-1}) in $\mathcal{T}$ contains the edge $(a, b)^1$ for some x, and the equation $b \circ y = a$ has only one solution in y because only the triangle (b^0, a^1, y^{-1}) in $\mathcal{T}$ contains the edge $(a, b)^{-1}$ for some y. Hence $(Q, \circ)$ is a commutative and idempotent quasigroup.

In the other direction, let $(Q, \circ)$ be a commutative and idempotent quasigroup. Define $\mathcal{T} = \{(a^0, c^1, b^{-1}) | a, b \in Q \text{ and } a \circ b = c\}$. Every triangle in this set is well-defined because $(a^0, c^1, b^{-1}) = (b^0, c^1, a^{-1})$. Let a, b be arbitrarily chosen elements in Q, $(a, b)^0$ belongs only to the triangle (a^0, c^1, b^{-1}) for some $c \in Q$ because $\circ$ is a well-defined binary operation. Then $(a, b)^1$ belongs only to the triangle (a^0, b^1, x^{-1}) where x is the unique solution to the equation $a \circ x = b$; and $(a, b)^{-1}$ belongs only to the triangle (b^0, a^1, y^{-1}) where y is the unique solution to the equation $b \circ y = a$. $\mathcal{T}$ is 3-balanced, and it is a Bose triangulation. ∎

If we take a commutative and idempotent quasigroup $(Q, \circ)$ of order v, build from it the Bose triangulation $\mathcal{T}$ given by Proposition 2.2 and finally build from $\mathcal{T}$ the STS($3v$) given by Theorem 2.1, then the resulting STS is the same as that obtained from $(Q, \circ)$ by using Bose's method directly. Bose triangulations provide only one way to find 3-balanced triangulations of $3\overline{K}_v$, but there are others. There are many possibilities, but we are interested in those 3-balanced triangulations with additional algebraic structure.

An *uniform triangulation* of $3\overline{K}_v$ is a 3-balanced triangulation of $3\overline{K}_v$ such that each of its triangles can be expressed as (a^0, b^1, c^{-1}) or (a^0, b^{-1}, c^1) for appropriate elements $a, b, c \in \{0, \ldots, v-1\}$. Triangles of the first type are *positive* and those of the second type *negative*. A positive triangle cannot be expressed as a negative one, nor vice versa. A Bose triangulation does not permit the mixture of positive and negative triangles, but in an uniform triangulation we admit this possibility. Look the following uniform triangulation of $3\overline{K}_v$ for $v = 7$, graphically represented in Figure 1:

$$\begin{aligned} \mathcal{T}_7 \ = \{ \ & \{0^0, 1^1, 2^{-1}\}, \{4^0, 1^{-1}, 0^1\}, \{6^0, 1^1, 4^{-1}\}, \{1^0, 6^1, 4^{-1}\}, \{4^0, 6^{-1}, 2^1\}, \\ & \{2^0, 6^1, 3^{-1}\}, \{2^0, 3^1, 6^{-1}\}, \{1^0, 3^{-1}, 2^1\}, \{5^0, 3^1, 1^{-1}\}, \{3^0, 5^1, 1^{-1}\}, \end{aligned}$$

$\{1^0, 5^{-1}, 6^1\}, \{6^0, 5^1, 0^{-1}\}, \{6^0, 0^1, 5^{-1}\}, \{3^0, 0^{-1}, 6^1\}, \{4^0, 0^1, 3^{-1}\},$
$\{0^0, 4^1, 3^{-1}\}, \{3^0, 4^{-1}, 5^1\}, \{5^0, 4^1, 2^{-1}\}, \{5^0, 2^1, 4^{-1}\}, \{0^0, 2^{-1}, 5^1\},$
$\{0^0, 2^1, 1^{-1}\}\}$

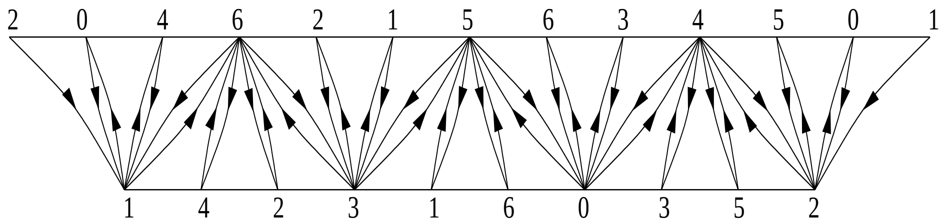

Figure 1: A uniform triangulation of $3\overline{K_7}$

When this triangulation is used in the construction of Theorem 2.1 we get an STS(21) isomorphic to the following, reading columns as triples:

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0000000000111111112222222222333333444444555555666667777888899aabccdd
13579bdfhj3469acfgi345678abe678begi5689abd789abc79beg9aef9abfcgceedhfh
2468acegik578bdehjk9fidcjgkhadcfkhjecgkhjikhdfgjijhfkhbgjekdifiijigkkj
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A direct analysis shows that it is anti-Pasch. It is well known (see [4]) that Bose's method does not produce an anti-Pasch STS(21), so our extension is not trivial.

3 3-tri algebras

In the same way that Bose's method can be formulated in terms of commutative and idempotent quasigroups, the construction given in Theorem 2.1 can be stated by using 3-tri algebras, algebraic structures that generalize quasigroups.

A *3-tri algebra*[1] (read this as *three triangulation algebra*) of order $v > 0$ is a pair $\Upsilon = (C, \circ)$ where C is a set with cardinality v and $\circ$ is a binary, closed, commutative and idempotent operation over C such that for every pair of distinct elements $a, b \in C$ the equations

$$a \circ x = b \tag{1}$$

$$b \circ y = a \tag{2}$$

with unknowns x and y, satisfy one and only one of the conditions:

[1] The selection of the name "3-tri algebra" was difficult. Certainly these structures are a weakening of quasigroups, so a name made of some prefix like "near", "half", "meta" or something similar followed by the word "quasigroup" could be better. However these names do not make clear that the source of 3-tri algebras are the 3-balanced triangulations. Other balanced partitions of the edges in a k-oriented graph, for some appropriate values of k, could be defined, probably some of them produce new algebras useful in design theory. The advantage of the name "3-tri algebra" is that it could be easily generalized with a clear meaning in this context.

1. There are exactly two solutions for x and none for y.
2. There are exactly two solutions for y and none for x.
3. There is exactly one solution for x and one for y.

Every commutative and idempotent quasigroup is a 3-tri algebra. One example of 3-tri algebra which is not a quasigroup is the pair $(\{0,\dots,6\},\circ)$ where $\circ$ is the operation shown in Figure 2. This is the 3-tri algebra used to generate the STS(21) given in Section 2.

$\circ$	0	1	2	3	4	5	6
0	0	2	1	4	1	2	5
1	2	1	3	5	6	3	5
2	1	3	2	6	6	4	3
3	4	5	6	3	0	4	0
4	1	6	6	0	4	2	1
5	2	3	4	4	2	5	0
6	5	5	3	0	1	0	6

Figure 2: Multiplication table of a 3-tri algebra

The multiplication table of a 3-tri algebra has a structure similar to that of a uniform square. However, an element can appear twice (at most) in a row; an element j does not appear in a row i if and only if i appears twice in the row j. Any idempotent and symmetric matrix with this property corresponds to a 3-tri algebra.

4 3-tri algebras and 2-$(v,3,3)$ designs

Our main interest in 3-tri algebras is their capacity to generalize Bose's method. However, as we show here, they have a strong link with 2-$(v,3,3)$ designs. Let $\Upsilon=(\{0,\dots,v-1\},\circ)$ be a 3-tri algebra. For every unordered pair $\{i,j\}$ of different elements in $\{0,\dots,v-1\}$, the set $T_{\Upsilon,\{i,j\}} \stackrel{\text{def}}{=} \{i,j,i\circ j\}$ (or $T_{\{i,j\}}$ when there is no confusion with the 3-tri algebra) is the *triple induced by i and j in Υ*. The set $\mathcal{T}_\Upsilon \stackrel{\text{def}}{=} \{T_{\{i,j\}} | \{i,j\}\subset\{0,\dots,v-1\}, i\neq j\}$ is the *set of triples induced by Υ*.

Let $\Upsilon=(\{0,1,\dots,7\},\circ)$ be the 3-tri algebra with the operation in Figure 2, then

$$\begin{aligned}
\mathcal{T}_\Upsilon \ = \{ \ & T_{\{0,1\}}=\{0,1,2\}, T_{\{0,2\}}=\{0,2,1\}, T_{\{0,3\}}=\{0,3,4\}, T_{\{0,4\}}=\{0,4,1\},\\
& T_{\{0,5\}}=\{0,5,2\}, T_{\{0,6\}}=\{0,6,5\}, T_{\{1,2\}}=\{1,2,3\}, T_{\{1,3\}}=\{1,3,5\},\\
& T_{\{1,4\}}=\{1,4,6\}, T_{\{1,5\}}=\{1,5,3\}, T_{\{1,6\}}=\{1,6,5\}, T_{\{2,3\}}=\{2,3,6\},\\
& T_{\{2,4\}}=\{2,4,6\}, T_{\{2,5\}}=\{2,5,4\}, T_{\{2,6\}}=\{2,6,3\}, T_{\{3,4\}}=\{3,4,0\},\\
& T_{\{3,5\}}=\{3,5,4\}, T_{\{3,6\}}=\{3,6,0\}, T_{\{4,5\}}=\{4,5,2\}, T_{\{4,6\}}=\{4,6,1\},\\
& T_{\{5,6\}}=\{5,6,0\}\}
\end{aligned}$$

As we can see it is a $2-(7,3,3)$ design, and in fact we have the following general result.

Proposition 4.1 *For any 3-tri algebra Υ of order v, $\mathcal{T}_\Upsilon$ is a 2-$(v,3,3)$ design.*

Proof: Every pair of distinct elements $a, b \in \{0, \ldots, v-1\}$ belongs to exactly three different triples in $\mathcal{T}_\Upsilon$. One is $T_{\{a,b\}}$, and the other two are:

Case 1: $T_{\{a,x_1\}}$ and $T_{\{a,x_2\}}$ where x_1 and x_2 are the two solutions to equation (1), or

Case 2: $T_{\{b,y_1\}}$ and $T_{\{b,y_2\}}$ where y_1 and y_2 are the two solutions to equation (2), or

Case 3: $T_{\{a,x_1\}}$ and $T_{\{b,y_1\}}$ where x_1 and y_1 are the solutions to equations (1) and (2). ■

$\mathcal{T}_\Upsilon$ is also called the *2-$(v,3,3)$ design induced by* Υ. Proposition 4.1 is a generalization of the well known fact (see [2], for example) that an idempotent and commutative quasigroup can be used to produce a 2-$(v,3,3)$ design. A converse is valid for 3-tri algebras:

Proposition 4.2 *Every 2-$(v,3,3)$ design generates a family of 3-tri algebras.*

Proof: Let $(\{0,\ldots,v-1\},\mathcal{T})$ be a 2-$(v,3,3)$ design. Let $G_\mathcal{T}$ be the bipartite graph with bipartition $V_1 = \{\{a,b\} | a \neq b, a, b \in \{0,\ldots,v-1\}\}$ and $V_2 = \mathcal{T}$, two vertices $\{i,j\} \in V_1$ and $T \in V_2$ being joined by an edge if and only if $\{i,j\} \subset T$. Then $G_\mathcal{T}$ is a 3-regular graph. We establish that each of its perfect matchings produces a 3-tri algebra of order v.

Let $M \subset E(G)$ be one such matching. We use the notation $M(i,j) = \{i,j,k\}$ if and only if $(\{i,j\},\{i,j,k\}) \in M$. Define a binary operation $\circ_M$ on $\{0,\ldots,v-1\}$ by

$$i \circ_M j = \begin{cases} k & \text{if } i \neq j \text{ and } M(i,j) = \{i,j,k\} \\ i & \text{if } i = j \end{cases}$$

Every set $\{a,b\} \in V_1$ is contained in three and only three triples in $\mathcal{T}$, so there exist two different elements c and d satisfying one of the following:

Case 1: $\{a,b\}$ belongs simultaneously to $M(a,b)$, $M(a,c) = \{a,b,c\}$ and $M(a,d) = \{a,b,d\}$.

Case 2: $\{a,b\}$ belongs simultaneously to $M(a,b)$, $M(b,c) = \{a,b,c\}$ and $M(b,d) = \{a,b,d\}$.

Case 3: $\{a,b\}$ belongs simultaneously to $M(a,b)$, $M(a,c) = \{a,b,c\}$ and $M(b,d) = \{a,b,d\}$.

The solutions for x and y to the equations $a \circ_M x = b$ and $b \circ_M y = a$ are as follows. In Case 1, c and d are solutions in x and y has no solution. In Case 2, c and d are solutions in y and x has no solution. Finally in Case 3, c is a solution in x and d a solution in y. Then $\circ_M$ is a commutative and idempotent binary operation. We conclude that $(\{0,\ldots,v-1\},\circ_M)$ is a 3-tri algebra produced from M. ■

5 Uniform triangulations and 3-tri algebras

As we saw in Theorem 2.1, the generalization of Bose's construction rests on our ability to find 3-balanced triangulations of $3\overline{K}_v$. The 3-tri algebras form an intermediate step between 3-balanced triangulations and quasigroups. In fact, 3-tri algebras of order v are 'almost' equivalent to uniform triangulations of $3\overline{K}_v$.

Proposition 5.1 *There exist a one to one correspondence between the set of uniform triangulations of $3\overline{K}_v$ and the set of 3-tri algebras of order v.*

Proof: Let $\mathcal{U}$ be a uniform triangulation of $3\overline{K}_v$. We build the 3-tri algebra $\Upsilon_{\mathcal{U}} = (\{0,\ldots,v-1\}, \circ_{\mathcal{U}})$ where $i \circ_{\mathcal{U}} j \stackrel{\text{def}}{=} k$ if and only if one of the following three conditions is satisfied:

1. $i = j = k$.
2. $(i^0, k^1, j^{-1}) \in \mathcal{U}$.
3. $(i^0, k^{-1}, j^1) \in \mathcal{U}$.

Then $\circ_{\mathcal{U}}$ is a commutative and idempotent binary operation. On the other hand, if a, b are different elements in $\{0,\ldots,v-1\}$, then $(a,b)^0 \in T_0$, $(a,b)^1 \in T_1$ and $(a,b)^{-1} \in T_{-1}$ where T_0, T_1 and T_{-1} are 3-different triangles in $\mathcal{U}$. There exist two different elements $c, d \in \{0,\ldots,v-1\}$ such that only one of the following cases is satisfied:

Case 1: $T_1 = (a^0, b^1, c^{-1})$ and $T_{-1} = (a^0, b^{-1}, d^1)$.

Case 2: $T_1 = (b^0, a^{-1}, c^1)$ and $T_{-1} = (b^0, a^1, d^{-1})$.

Case 3: $T_1 = (a^0, b^1, c^{-1})$ and $T_{-1} = (b^0, a^1, d^{-1})$.

The solutions in x and y to the equations $a \circ_{\mathcal{U}} x = b$ and $b \circ_{\mathcal{U}} y = a$ are as follows. In Case 1, c and d are solutions in x, and y has no solution. In Case 2, c and d are solutions in y, and x has no solution. Finally in Case 3, c is a solution in x and d a solution in y. We conclude that $\Upsilon_{\mathcal{U}}$ is a 3-tri algebra. ∎

The converse of this proposition does not hold. Only some 3-tri algebras, to be characterized, produce uniform 3-tri algebras of $3\overline{K}_v$. Let $\Upsilon = (\{0,\ldots,v-1\}, \circ)$ be a 3-tri algebra of order v. The *Bose graph* of Υ, denoted B_Υ, is a graph with the triples in $\mathcal{T}_\Upsilon$ as vertices, two vertices $T_{\{i_1,j_1\}}$ and $T_{\{i_2,j_2\}}$ being joined by an edge if and only if the corresponding triples share a pair $\{i,j\}$ such that $\{i,j\} \neq \{i_1,j_1\}$ and $\{i,j\} \neq \{i_2,j_2\}$. The same idea can be expressed in terms of Υ by saying that $T_{\{i_1,j_1\}}$ and $T_{\{i_2,j_2\}}$ are adjacent if one of the following conditions is true (as shown in Figure 3):

Condition 1: $j_1 = i_2 \circ j_2$ and $i_2 = i_1 \circ j_1$.

Condition 2: $j_1 = i_2$ and $i_1 \circ j_1 = i_2 \circ j_2$.

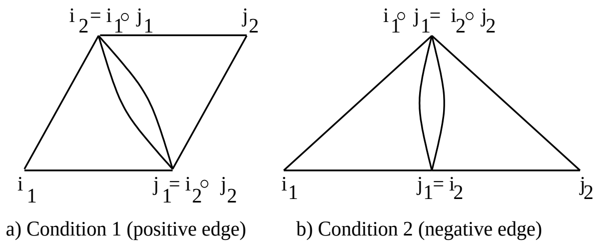

Figure 3: Adjacencies in B_Υ

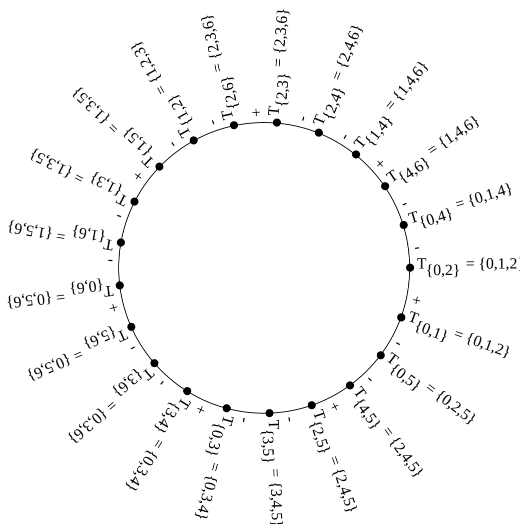

Figure 4: The Bose graph of the 3-tri algebra in Section 3

An edge in B_Υ is *positive* if it satisfies Condition 1; otherwise it is *negative*. Figure 4 depicts the Bose graph of the 3-tri algebra in Figure 2. In this case the graph is a cycle.

Lemma 5.2 *If* $\Upsilon = (\{0,\ldots,v-1\},\circ)$ *is a 3-tri algebra, then* B_Υ *is a 2-regular simple graph.*

Proof: A triple $T_{\{a,b\}} = \{a,b,c\}$ in $V(B_\Upsilon) = \mathcal{T}_\Upsilon$ is only adjacent to triples containing $\{a,c\}$ and $\{b,c\}$. Since $\mathcal{T}_\Upsilon$ is a 2-$(v,3,3)$ design, other than $T_{\{a,b\}}$ there are only two triples containing $\{a,c\}$. One is $T_{\{a,c\}}$, but it is not adjacent to $T_{\{a,b\}}$. The other is one of the following two possibilities:

Case 1: $T_{\{a,x\}}$ where $a \circ x = c$ and $x \neq b$; or

Case 2: $T_{\{c,y\}}$ where $c \circ y = a$.

The case depends upon the solutions of the equations $a \circ x = c$ and $c \circ y = a$. In either situation such a triple is the only one adjacent to $T_{\{a,b\}}$ which contains $\{a,c\}$. Similarly $T_{\{a,b\}}$ is also adjacent to only one of the following triples containing $\{b,c\}$:

Case 1': $T_{\{b,x'\}}$ where $b \circ x' = c$ and $x' \neq a$; or

Case 2': $T_{\{c,y'\}}$ where $c \circ y' = b$.

The triple from Cases 1 and 2, $T_{\{a,b\}}$, and the triple from Cases 1' and 2' are different, so $T_{\{a,b\}}$ has degree two and its incident edges are neither loops nor parallel edges in B_Υ. We conclude that this is a 2-regular simple graph. ■

Let Υ be a 3-tri algebra of order v. Any function $\sigma : \{\{i,j\} | i \neq j \text{ and } i,j \in \{0,\ldots,v-1\}\} \to \{+,-\}$ such that for every edge $e = (T_{\{a,b\}}, T_{\{c,d\}})$ in $E(B_\Upsilon)$ $\sigma(a,b) = \sigma(c,d)$ if and only if e is positive is a *signing* of Υ. If Υ has at least one signing it is *signable*; otherwise it is *unsignable*.

Lemma 5.3 *A 3-tri algebra* Υ *is signable if and only if every cycle in* B_Υ *has an even number of negative edges.*

Proof: Let σ be a signing of Υ and let $P = T_{\{a_0,b_0\}}, \ldots, T_{\{a_k,b_k\}}$ be a path in B_Υ, $\sigma(a_k,b_k) = \sigma(a_0,b_0)(-1)^n$ where n is the number of negative edges in P; so σ is well defined if and only if the number of negative edges in every cycle of B_Υ is even. ■

The multiplication table of an unsignable 3-tri algebra is given in Figure 5. It is unsignable because its Bose graph contains the cycle $(T_{\{4,5\}}, T_{\{5,6\}}, T_{\{4,6\}})$ in which the three edges are negative. Let $\Upsilon = (\{0,\ldots,v-1\},\circ)$ be a signable 3-tri algebra, and let σ be one of its signings. For every pair a,b of different elements in $\{0,\ldots,v-1\}$, the 3-oriented cycle $\overline{T}_{\Upsilon,\sigma,a,b} \stackrel{\text{def}}{=} (a^0, (a\circ b)^{\sigma(a,b)}, b^{-\sigma(a,b)})$ (or $\overline{T}_{a,b}$ when there is no confusion with Υ and σ) is the *3-oriented cycle induced by* Υ, σ, a *and* b. The set $\overline{\mathcal{T}}_{\Upsilon,\sigma} \stackrel{\text{def}}{=} \{\overline{T}_{a,b} | a \neq b \text{ and } a,b \in \{0,\ldots,v-1\}\}$ is the *set of 3-cycles induced by* Υ and σ. The sets $\mathcal{T}_\Upsilon$ and $\overline{\mathcal{T}}_{\Upsilon,\sigma}$ are essentially the same, but in the latter we have chosen orientations.

$\circ$	0	1	2	3	4	5	6
0	0	2	3	1	3	2	1
1	2	1	4	5	5	6	2
2	3	4	2	6	5	3	4
3	1	5	6	3	1	6	4
4	3	5	5	1	4	0	0
5	2	6	3	6	0	5	0
6	1	2	4	4	0	0	6

Figure 5: Multiplication table of an unsignable 3-tri algebra

Proposition 5.4 *If $\Upsilon = (\{0,\ldots,v-1\},\circ)$ is a signable 3-tri algebra of order v and σ is one of its signings, then $\overline{\mathcal{T}}_{\Upsilon,\sigma}$ is a uniform triangulation of $3\overline{K}_v$.*

Proof: Let a, b be two different elements in $\{0,\ldots,v-1\}$. We establish that each of the edges $(a,b)^0, (a,b)^1$ and $(a,b)^{-1}$ belongs to exactly one 3-cycle in $\overline{\mathcal{T}}_{\Upsilon,\sigma}$. Evidently $(a,b)^0$ belongs only to $\overline{T}_{a,b}$. Now we have three possibilities:

Case 1: The equation $a \circ x = b$ has two solutions in x, say c and d. $(T_{\{c,a\}}, T_{\{a,d\}})$ is a negative edge in B_Υ, so $\sigma(a,d) = -\sigma(a,c)$, and thus $(a,b)^{\sigma(a,c)}$ belongs to $\overline{T}_{a,c} = (a^0, b^{\sigma(a,c)}, c^{-\sigma(a,c)})$ and $(a,b)^{-\sigma(a,c)}$ belongs to $\overline{T}_{a,d} = (a^0, b^{-\sigma(a,c)}, c^{\sigma(a,c)})$. No other 3-cycle in $\overline{\mathcal{T}}_{\Upsilon,\sigma}$ contains $\{a,b\}$.

Case 2: The equation $b \circ y = a$ has two solutions in y. This is similar to Case 1.

Case 3: The equations $a \circ x = b$ and $b \circ y = a$ have one solution in x and one in y, say $x = c$ and $y = d$. $(T_{\{c,a\}}, T_{\{b,d\}})$ is a positive edge in B_Υ, so $\sigma(b,d) = \sigma(a,c)$, and thus $(a,b)^{\sigma(a,c)}$ belongs to $\overline{T}_{a,c} = (a^0, b^{\sigma(a,c)}, c^{-\sigma(a,c)})$ and $(a,b)^{-\sigma(a,c)}$ belongs to $\overline{T}_{b,d} = (b^0, a^{\sigma(b,d)}, c^{-\sigma(b,d)})$. No other 3-cycle in $\overline{\mathcal{T}}_{\Upsilon,\sigma}$ contains $\{a,b\}$.

Since all 3-cycles in $\overline{\mathcal{T}}_{\Upsilon,\sigma}$ have the form of a uniform triangulation we conclude that it is a uniform triangulation of $3\overline{K}_v$. ∎

6 The Skolem method

We use the idea of Theorem 2.1 to generalize the Skolem method (see [2], for example). Let v be a positive even integer, say $v = 2n$. Denote by $3\overline{K}_v'$ the graph $3\overline{K}_v - \{(a,n+a)^{-1} | a \in \{0,\ldots,n-1\}\} \bigcup \{(n+a,n+a)^1 | a \in \{0,\ldots,n\}\}$. Then $3\overline{K}_v'$ is not simple, since we have replaced a perfect matching of negative edges in $3\overline{K}_v$ by positive loops on the vertices $n, n+1, \ldots, 2n-1$.

Theorem 6.1 *Every 3-balanced triangulation of $3\overline{K}_v'$ yields an STS$(3v+1)$.*

Proof: Let $\mathcal{T}$ be a 3-balanced triangulation of $3\overline{K}'_v$. Let us define:

$$X = \{(a,i) | a \in \{0,\dots,n-1\} \text{ and } i \in \{0,1,2\}\} \bigcup \{\infty\},$$

$$\mathcal{A}_\infty = \{\{(a,(i+1) \bmod 3),(a+n,i),\infty\} | a = 0,1,\dots,n-1\},$$

$$\mathcal{A}_1 = \{\{(a,0),(a,1),(a,2)\} | a = 0,1,\dots,n-1$$

and for each $T = (a^{\theta_a}, b^{\theta_b}, c^{\theta_c}) \in \mathcal{T}$

$$\mathcal{A}_T = \{\{(a,j),(b,(j+\theta_b) \bmod 3),(c,(j+\theta_b+\theta_c) \bmod 3)\} | j = 0,1,2\}.$$

In the same manner as in the proof of Theorem 2.1, $(X,\mathcal{A})$ with $\mathcal{A} = \mathcal{A}_\infty \bigcup \mathcal{A}_1 \bigcup (\bigcup_{T\in\mathcal{T}} \mathcal{A}_T)$ is an STS$(3v)$.

It is possible to develop an algebraic structure similar to 3-tri algebras to find 3-balanced triangulations of $3\overline{K}'_v$. However the resulting structure does not share the nice properties of 3-tri algebras and we prefer to omit it.

7 Conclusions

Theorem 2.1 gives us a technique to generalize one of the most important methods to construct Steiner triple systems. The real potential of this construction depends upon our ability to generate 3-balanced triangulations of $3\overline{K}_v$. The 3-tri algebras give some solutions to this problem, but they are not the only possibility. The general problem of determining all 3-balanced triangulations of $3\overline{K}_v$ remains open.

The construction of signable 3-tri algebras is not easy, we have studied some methods which are reported in [3]. In this work we showed that it is possible to generate 3-tri algebras appropriate for the construction of anti-Pasch Stainer triple systems. These methods are based on an interesting application of the eight queens problem.

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References

[1] R. C. Bose. On the construction of balanced incomplete block designs, *Ann. Eugenics* **9** (1939), 353-399.

[2] C.J. Colbourn and A. Rosa, *Triple Systems*, Oxford University Press, 1999.

[3] C.J. Colbourn and F. Sagols, NS1D0 sequences, 3-triangulations and anti-Pasch STSs. Accepted in *Ars Combinat.* in march 2000.

[4] M.J. Grannell, T.S. Griggs, and J.S. Phelan, A new look at an old construction for Steiner triple systems, *Ars Combinat.* 25A (1988), 55-60.