Hamiltonian Representation of Vox-solids

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Abstract

We present here an extended abstract of the work [7]. Our objective is the study of vox-solids and their applications. A vox-solid is a solid digitized into unitary cubes whose boundary is a surface; the face adjacency graph embedding of a vox-solid is the graph whose vertices are the voxel's boundary faces, and the edges relate pairs of boundary faces sharing exactly one side. Our main conjecture says that every face adjacency graph is Hamiltonian.

1 Introduction

References to definitions, propositions, theorems and so on correspond to those used in [7].

The boundary representation of a region in the plane consists of a traversing around its boundary [2]; for this, the given region is digitized into a finite union of pixels (unit squares.) The main motivation of this work was generalizing this representation to vox-solids, which are solids digitized into a finite union of voxels (unit cubes), by also traversing their boundaries. In our representation we associate with each vox-solid a graph whose vertices correspond with the faces on its boundary, and whose edges indicate the adjacency relationship between faces. For a given vox-solid, such a graph is called the *face adjacency graph* of the vox-solid.

The problem that we study consists in finding Hamiltonian cycles on the face adjacency graph of vox-solids. From a combinatorial point of view this is a very interesting problem since it is related with the characterization of 4-regular and 4-connected Hamiltonian graphs. Our main conjecture states that every face adjacency graph is Hamiltonian.

This conjecture is interesting because, if true, it would give us an important family of Hamiltonian 4-regular graphs on every oriented surface, in contrast to seeking a complete characterization of all the Hamiltonian 4-regular graphs on a specific surface. Important attempts have been made for the later which in general is a very difficult problem; on the other hand, the approach followed in this work, by further introducing the restriction of voxelability seems to be enough to characterize a very wide family of Hamiltonian 4-regular graphs. The problem is extremely challenging. We believe that any method developed to solve it will produce new interesting theoretical results and combinatorial structures.

But this conjecture is not only important from a theoretical point of view. In Computer Science it has applications in representing 3-dimensional solids by considering the Hamiltonian cycle on their face adjacency graph as a chain of symbols in an alphabet describing the boundary. Another application is the design of interconnection topologies in network systems and parallel computer architectures. In conclusion, the study of this conjecture is important from the Combinatorics and the Computer Science points of view.

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2 Vox-solids

In order to define and characterize "vox-solids", we use two different topological ways to view 3-space. The first one involves the use of *gratings* which are subdivisions into cubes of the Euclidean 3-space by planes parallel to the coordinate axes. All the elements in the subdivision are *voxels*, that is, identical cubes with edges of length 1. We always work with the grating $\mathcal{G}$ whose voxels have integer barycenters, in this way we understand by *the coordinates of a voxel* the coordinates of its barycenter. Some useful subsets of $\mathcal{G}$ are the 0,1,2 or 3-*simplexes* which respectively are voxel corners, voxel edges, voxel faces and voxels; we say that a k-simplex has k-*dimension*.

A finite union of simplexes with the same dimension k is a k-*chain*, $0 \leq k \leq 3$; no chain contains simplexes of different dimensions. The *boundary* $\partial(C)$ of a k-chain C is a $k-1$-chain containing the set of $k-1$-simplexes that belong to an odd number of k-simplexes in C.

In these terms a *vox-solid* $\mathcal{V}$ is a connected 3-chain whose boundary is a surface (**definition 2.4.1**.) Some examples appear in figure 1

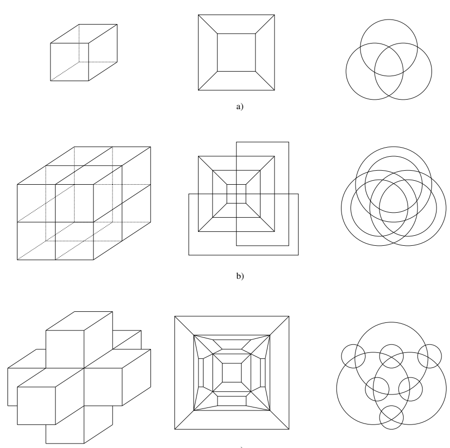

Figure 1: Vox-solids, their tilings and face adjacency graphs

The second topological interpretation of the space that we use comes from *Digital Topology* (see [5]), this field deals with digital pictures which are a finite subset B of the integer lattice in R^3 (or in R^2), and two adjacency relations. B is called the *black points set* and its complement the *white points set*; the first adjacency relation establishes when two black points are adjacent and the second one when two white points or a black and a white points are adjacent. The adjacency relations are chosen in such a way that the resulting structure defines a topological space.

The voxels' barycenters in a finite 3-chain determine the black points set of a unique 3-dimensional digital picture; conversely, the black points of a 3-dimensional digital picture define the voxels' barycenters of a unique 3-chain. Not all 3-chains are vox-solids, this is only true when the digital picture associated to a 3-chain has one black and one white components, and it does not contain the configurations pictured in figure 2 (**theorem 2.4.2**.)

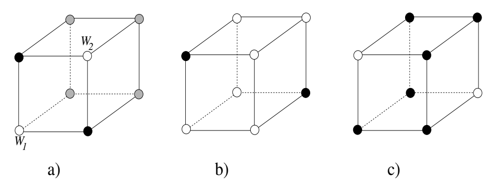

Figure 2: Configurations in theorem 2.4.2. The gray points in a) may be black or white

One special type of vox-solids are the *step vox-solids* in which is possible to order its voxels into a sequence $V_1, \ldots, V_n$ in such a way that for each i, $(1 \leq i \leq n)$ the chain $\{V_1, \ldots, V_i\}$ is a vox-solid (**definition 2.5.1**.) On step vox-solids it is possible to use inductive arguments based on voxel by voxel constructions as a technique of proof; unfortunately not every vox-solid is a step vox-solid, one counter-example appears in figure 3. An interesting open problem is the characterization of step vox-solids. (**problem 2.5.1**.)

3 Main conjecture

Given a topological space S we denote by $C(S)$ the set of connected components of S. A *surface embedding* (see [3] and [4]) Ψ is defined to be a triple (Σ, U, V) where Σ is a surface, U is a closed subset of Σ and V is a finite subset of U such that $C(U-V)$ is a finite set of homeomorphic copies of

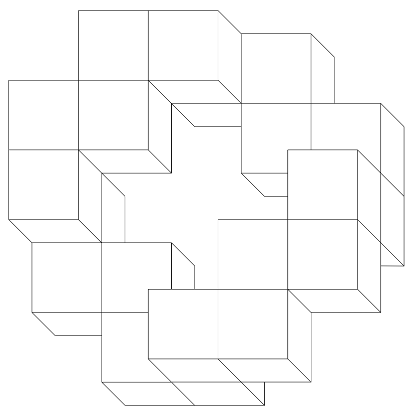

Figure 3: a) A toroidal non step vox-solid $\mathcal{V}$. b) $\mathcal{T}(\mathcal{V})$

the open unit interval (0,1). We denote by $\Sigma(\Psi)$ the surface in which Ψ is embedded, by $V(\Psi)$ the *vertex set* V of Ψ, by $E(\Psi)$ the *edge-set* $C(U-V)$ of Ψ and by $F(\Psi)$ the *face set* $C(\Sigma - U)$ of Ψ.

Given a 3-chain T the *face adjacency graph* of T is the graph τ_T having as vertices the set of 2-simplexes in $\partial(T)$, two vertices F_1, and F_2 being joined by an edge if and only if F_1 and F_2 have just a 1-simplex in common (**definition 3.2.1**.) The *tiling graph* of T, denoted by τ_T^*, has as vertices the set of 0-simplexes in $\partial(T)$ and two vertices P_1, P_2 form an edge if and only if P_1 and P_2 are the ends of a 1-simplex in $\partial(T)$ (**definition 3.2.2**.) A graph G is *voxelable* if there exists a vox-solid $\mathcal{V}$ such that G is isomorphic to $\tau_{\mathcal{V}}$. (**definition 3.2.4**.) A surface embedding Ψ is said to be voxelable if $G(\Psi)$ is voxelable (**definition 3.2.5**.) The characterization of voxelable graphs is a very difficult problem which we leave open (**problem 3.2.1**); we give then a list of necessary conditions, but still we don't have a sufficient condition.

Let $\mathcal{V}$ be a vox-solid, the *natural tiling embedding of* $\mathcal{V}$ is the embedding

$$\Psi_{\mathcal{V}}^* \stackrel{\text{def}}{=} (\partial(\mathcal{V}), \{S|S \text{ is a 1-simplex in } \partial(\mathcal{V})\}, \{T|T \text{ is a 0-simplex in } \partial(\mathcal{V})\})$$

The dual of this embedding is denoted by $\Psi_{\mathcal{V}}$ and is called the *natural face adjacency graph embedding of* $\mathcal{V}$.

Notice that two different vertices in $G(\Psi_{\mathcal{V}}^*)$ are adjacent if and only if they are the endpoints of a 1-simplex in $\partial(\mathcal{V})$, this and the adjacency relation of $\tau_{\mathcal{V}}^*$ are the same, in consequence these graphs are isomorphic. Similarly $G(\Psi_{\mathcal{V}})$ is isomorphic to $\tau_{\mathcal{V}}$.

On the other hand, we use a result from Tutte which states that every 4-connected planar graph is Hamiltonian (**theorem 3.3.3**) [10], this result together with theorems **3.3.1** and **3.1.4** implies that the face adjacency graph of a spherical vox-solid is always Hamiltonian (**corollary 3.3.4**.) From an algorithmic point of view Chiba and Nishizeky proved that a Hamiltonian cycle can be found in 4-connected planar graphs in linear time [1]

Theorem 3.3.1 extends the conditions known on embeddings in order to have a 4-connected graph embedded on a surface with 4-representativity (see [6]); theorem 3.1.4 states that every face adjacency graph is 4-connected.

A result similar to theorem 3.3.3 was conjectured for toroidal graphs but remains open yet. Robin Thomas and Xingxing Yu [9] proved that 5-connected toroidal graphs are Hamiltonian, the same authors proved that 4-connected projective-planar graphs are Hamiltonian [8].

We finish with the statement of the main conjecture in our work: Every face adjacency graph is Hamiltonian (**conjecture 3.6.1**.)

4 Properties of voxelable graphs

In our work we give additional necessary conditions for a graph to be voxelable, we also give an algorithm to reconstruct certain vox-solids from their natural face adjacency graph embedding.

Let Ψ be an embedding with $G(\Psi)$ 4-regular and $\Sigma(\Psi)$ oriented. The *straight decomposition* $\mathcal{K}_\Psi$ of Ψ is the partition of edges in $U(\Psi)$ into edge disjoint cycles $C_1, \ldots, C_k$ in such a way that each edge is traversed exactly once by these cycles and such that for each vertex W of $V(\Psi)$ if e_1, e_2, e_3 and e_4 are the edges incident with W in cyclic order (with regard to Ψ) then $e_1 W e_3$ are traversed consecutively (in one way or in the other) and similarly $e_2 W e_4$ are traversed consecutively (in one way or in the other.) This decomposition is unique up to the choice of the beginning vertex of the curves, up to reversing the curves and up to permuting the indices of $C_1, \ldots, C_k$.

When $\mathcal{V}$ is a vox-solid the cycles in the straight decomposition of $\Psi_{\mathcal{V}}$ are called *orthogonal cycles* (**definition 4.4.1**.) The most important result on this decomposition states that any voxelable graph G always admits a decomposition into three classes of concentric orthogonal cycles, such that every vertex is the intersection of two cycles in different classes (**theorem 4.2.1**.)

We use this results to construct, when it is possible, a vox-solid with face adjacency graph isomorphic to $G(\Psi)$ for a given embedding Ψ. The method takes as reference a fixed coordinate system Ω and seeks for a 2-chain whose 2-simplexes correspond with the vertices in $G(\Psi)$ and have the same adjacencies as those in the graph; an important role is played by the orthogonal cycles, which suggest us the types of faces with regard to Ω (up, down, left, right, front or back) that must be associated in this 2-chain. If the resulting 2-chain is the boundary of a vox-solid then we conclude that Ψ is voxelable.

5 Hamiltonicity of voxelable graphs

In this section we give new ways to see our problem and talk about the Hamiltonicity of certain families of voxelable graphs. For sake of clarity, we introduce some definitions.

Let G be a graph, a *2-factor* of G is a 2-regular spanning subgraph of G. If A and B are nonempty subsets of $V(G)$ then, $[A, B]$ denotes the set of edges in G with an end in A and the other in B. If G is 2-cellularly embedded into an oriented surface $\mathcal{S}$ then we denote by G^* the dual of G and by $[A, B]^*$ the set of dual edges in G^* associated to the edges in $[A, B]$.

The problems that we study in this section are:

Problem P1: Let Ψ be a 2-cellular embedding with $\Sigma(\Psi)$ oriented and $G(\Psi^*)$ 4-regular. Problem **P1** on Ψ consists in finding a set $\mathcal{A} \subset V(\Psi)$ such that:

1. Every face in Ψ has at least one vertex in $\mathcal{A}$.
2. Every face in Ψ has at least one vertex in $\overline{\mathcal{A}}$.
3. If a face in Ψ has just two vertices in $\mathcal{A}$ then both are adjacent in $G(\Psi)$.

Problem RP1: Similar to **P1** but condition 2 is replaced by "$G(\Psi)\langle\mathcal{A}\rangle$ is a tree".

Clearly a solution to **RP1** is a solution to **P1** but the converse in general is not true. A solution $\mathcal{A}$ to **P1** on an embedding Ψ with $G(\Psi)$ 4-regular and $\Sigma(\Psi)$ oriented generates a 2-factor on $G(\Psi)^*$, in fact it is $G(\Psi^*)\langle[\mathcal{A}, \overline{\mathcal{A}}]^*\rangle$ (**proposition 5.1.2**), but if $\mathcal{A}$ is a solution to **RP1** then this 2-factor is a Hamiltonian cycle (**proposition 5.1.4**.) The converse of the last is only true on the sphere.

In general finding a solution to **RP1** is a difficult problem which we leave open; in contrast, finding solutions to **P1** is very easy for the natural tiling embeddings of vox-solids, the basic idea consists in finding a solution $\mathcal{A}_0$ to **P1** on the natural tiling embedding of a vox-solid $\mathcal{V}_0$ formed by a single voxel, $\mathcal{A}_0$ is then transformed into a solution to **P1** for the natural tiling embedding of an arbitrary vox-solid $\mathcal{V}$. The precise method to do that transformation is given in proposition 5.2.1.

Under certain circumstances, whose study is the main subject of the following section, the solutions to **P1** found by the last method, can easily be transformed into solutions to **RP1**, or equivalently into Hamiltonian cycles for a face adjacency graph.

As an example of the way in which this strategy is applied, we introduce an infinite family of vox-solids whose face adjacency graph is always Hamiltonian:

Let G be a simple connected graph with edges $\{e_1, \ldots, e_m\}$. One *thickening* of G, is a graph G^T satisfying the following conditions:

1. For each vertex W in G, G^T contains a k-cycle C_W with $k = v(G : W)$[1]. If $e_{i_1}, e_{i_2}, \ldots, e_{i_k}$ are the edges in G incident to W, then we label the edges of C_W arbitrarily with one and only one of the symbols in the set $\{e_{i_1}(W),\ e_{i_2}(W),\ \ldots,\ e_{i_k}(W)\}$, and we label the vertices in C_W arbitrarily with one and only one of the symbols in the set $\{W^1, W^2, \ldots, W^k\}$.

2. For every i $(1 \leq i \leq m)$ suppose that $e_i = (U, W)$, $e_i(U) = (U^{j_1}, U^{j_2})$ and $e_i(W) =$

[1] The degree or valency of W in G.

(W^{k_1}, W^{k_2}), then G^T contains one of the set of edges $\{(U^{j_1}, W^{k_1}), (U^{j_2}, W^{k_2})\}$ or $\{(U^{j_1}, W^{k_2}), (U^{j_2}, W^{k_1})\}$ but not both. We denote by C_e the 4-cycle in G^T with vertices $U^{j_1}, W^{k_1}, W^{k_2}$ and U^{j_2}).

An easy but important result says that the thickening of a simple and connected graph is always Hamiltonian (**proposition 5.3.1**.)

On the other hand, let $\mathcal{V}$ be a vox-solid, divide every voxel in $\mathcal{V}$ into eight identical cubes and make them grow (preserving the proportions) until they become voxels again. Clearly we obtain a new vox-solid which we call the *refinement* of $\mathcal{V}$ and denote it by $\mathcal{V}^3$. A vox-solid is a *thick vox-solid* if and only if it is the refinement of another vox-solid (see figure 4.) The link between the refinement of a vox-solid and the thickening of a graph is given by the following result.

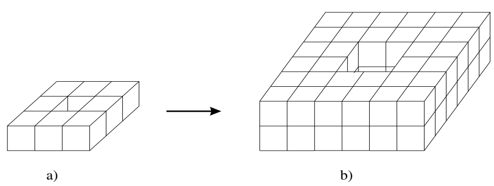

Figure 4: a) A vox-solid $\mathcal{V}$, b) $\mathcal{V}^3$

Let $\mathcal{V}$ be a vox-solid then $\tau_{\mathcal{V}^3}$ is the thickening of $\tau_{\mathcal{V}}$ (**lemma 5.3.3**.)

Having as a consequence that the thickening of a voxelable graph is a voxelable and Hamiltonian graph (**theorem 5.3.4**.)

This is important because one of our motivations was the generalization of Freeman's chains to 3-dimensions. These chains give a boundary description of a 2-dimensional digital picture by means of a chain formed by the consecutive edges in the boundary of the black components, the objective is to have a short description of the image (see [2].) Theorem 5.3.4 warranties that for every vox-solid we can always make a Hamiltonian description of its boundary if we work in its refinement; the price we pay is the increasing, by a constant factor, of the number of boundary elements (2-simplexes) needed in our representation. So, from the Computer Science point of view, this theorem solves the problem of the representation of the boundary of a vox-solid by a finite chain (the 3-dimensional version of Freeman's chains.) As discussed in our original work, we can find such a representation in linear time on the number of 2-simplexes in $\partial(\mathcal{V})$. In this way, theorem 5.3.4 is a satisfactory practical result.

6 Heuristic to find Hamiltonian cycles on the face adjacency graph of vox-solids

In chapter 6 of our original work we give an heuristic method to find Hamiltonian cycles on the face adjacency graph $\tau_{\mathcal{V}}$ of a vox-solid $\mathcal{V}$. Our method is divided into three phases. In the first one we find a partition of $\mathcal{V}$ in which all the elements are spherical vox-solids and at most one of them is non-spherical, then we find, if it is possible, (with the aid of corollary 3.3.4 and phases 2 and 3 of this method) Hamiltonian cycles on each sub-vox-solid which then are merged together to obtain a Hamiltonian cycle in $\tau_{\mathcal{V}}$. The second phase is applied only to non-spherical vox-solids which in certain way are of minimum size, the key idea consists in finding 2-factors on $\tau_{\mathcal{V}}$ which some times can be transformed into Hamiltonian cycles by making local reconfigurations. Finally, the third phase is similar to the second one but it works with more complex local reconfigurations.

7 Circular graphs and Hamiltonicity of 4-regular graphs

We give an alternative presentation to the problem of Hamiltonicity in face adjacency graphs in terms of circle graphs. It is a nice combinatorial formulation which unfortunately requires several advanced definition and results from graph theory and matroid theory that cannot be placed in this short space. We encourage the interested reader to see this material in the chapter 7 of the original work.

8 A new topology for processor interconnection in parallel computers

Interconnection networks are basic schemes to interconnect individual processors by links able to transmit binary data; applications of them are widely found in computer science, VLSI design, parallel computers, computer networks and telecommunication systems. The topological properties of an interconnection network determine its reliability, fault tolerance, mean transmission time and in general all its performance.

Interconnection network topologies are *static* and *dynamic*. In a static one, point-to-point links interconnect the network nodes in some fixed topology while in a dynamic the links can be switched. We study the static class.

An *interconnection network topology* is a graph whose vertices and edges represent processors and dedicated links between processors respectively. Almost all static network topologies studied in the literature have some degree of symmetry. Such a symmetric topology has many advantages: First it allows the network to be constructed from simple building blocks and expanded in a modular fashion. Second, regularity in the topology facilitates the use of simple routing algorithms. Third, it is easier to develop efficient computational algorithms for multiprocessors interconnected by a symmetric network. Finally, it makes the network easier to model and analyze. In a symmetric topology, a few simple rules are sufficient to specify the entire topology.

These reasons justify why the static network interconnection topologies used in practice have a "high" level of symmetry and regularity. Voxelable graphs are 4-regular with sub-families having different levels of symmetry, some of them are suitable as interconnection topologies.

9 Conclusions

Vox-solids and voxelable graphs opened a very interesting bridge between combinatorics, topological graph theory and computation. We think that this bridge can be extended to another areas in discrete mathematics and we can develop further applications in engineering. The problems left open are very challenging and could do important contributions in mathematics.

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