NS1D0 Sequences and Anti-Pasch Steiner Triple Systems

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Abstract

We present an algorithmic construction of anti-Pasch Steiner triple systems for orders congruent to 9 mod 12. This is a Bose-type method derived from a particular type of 3-triangulations generated from non-sum-one-difference-zero sequences (NS1D0 sequences). We introduce NS1D0 sequences and describe their basic properties; in particular we develop an equivalence between the problem of finding NS1D0 sequences and a variant of the n-queens problem. This equivalence, and an algebraic characterization of the NS1D0 sequences that produce anti-Pasch Steiner triple systems, form the basis of our algorithm.

1 Background

Let X be a finite set. A *set system* or *configuration* is a pair $(X, \mathcal{A})$, with $\mathcal{A} \subseteq 2^X$. The *order* of the set system is $|X|$. The elements of X are *points* and the elements of $\mathcal{A}$ are *blocks*. A t-(v, k, λ) design is a k-uniform set system $(X, \mathcal{A})$ of order v such that each t-subset of X is contained in precisely λ blocks of $\mathcal{A}$. A 2-$(v, 3, 1)$ design is a *Steiner triple system* of order v and is denoted by STS(v). See [2] for an extensive introduction to triple systems. A (k, ℓ)-*configuration* in an STS $(X, \mathcal{A})$ is a subset of ℓ blocks in $\mathcal{A}$ whose union is a k-element subset of X. The *Pasch configuration* or *quadrilateral* is the (6,4)-configuration on elements (say) a, b, c, d, e, f with blocks $\{a, b, c\}, \{a, d, e\}, \{f, d, b\}$ and $\{f, c, e\}$. An STS is *anti-Pasch* (or *quadrilateral-free*) if it does not contain the (6,4)-configuration. The existence of anti-Pasch Steiner triple systems has been the subject of numerous papers, and has only recently been settled [4]. Nevertheless, explicit constructions of such systems remains important, in part because more complex problems such as the existence of anti-Pasch resolvable Steiner triple systems remain open and of practical concern [1].

We introduce the concept of 3-triangulation and present without proof some relevant elementary properties. A detailed study of 3-triangulations containing proofs appears in [3].

Given an odd integer $n > 1$, a 3-triangulation is an edge-coloration Υ of K_n with colors $\{0, \ldots, n-1\}$ such that for each pair i, j of different elements in $\{0, \ldots, n-1\}$, one and only one of the following conditions is satisfied:

3TRIAN-1: Vertex i is incident with exactly two j-colored edges and vertex j is not incident with any i-colored edge.

3TRIAN-2: Vertex j is incident with exactly two i-colored edges and vertex i is not incident with any j-colored edge.

3TRIAN-3: Vertex i is incident with exactly one j-colored edge and vertex j is incident with exactly one i-colored edge.

Given a 3-triangulation Υ of order n, its *Bose graph*, denoted B_Υ, has the edges of K_n as vertices, and two of these vertices, say $e_1, e_2, e_1 \neq e_2$ are adjacent if and only if one of these two conditions is true:

BOSE-1: e_1 and e_2 are adjacent in K_n and $\Upsilon(e_1) = \Upsilon(e_2)$,

BOSE-2: $e_1 = (\Upsilon(e_2), i)$ and $e_2 = (\Upsilon(e_1), j)$ for some vertices i, j in K_n.

A BOSE-1 adjacency is *negative* and a BOSE-2 adjacency is *positive*.

For any 3-triangulation Υ, B_Υ is a 2-regular simple graph (Lemma 5.2 in [3].)

Let Υ be a 3-triangulation of order n. A function $\sigma : V(B_\Upsilon) \to \{+, -\}$ is a *signing* of Υ if, for any pair of adjacent vertices in B_Υ, say e_1 and e_2, $\sigma(e_1) = \sigma(e_2)$ if and only if the adjacency between e_1 and e_2 is positive. Υ is *signable* if there exists a signing of Υ.

A 3-triangulation Υ is signable if and only if each cycle in B_Υ has an even number of negative adjacencies (Lemma 5.3 in [3].)

Any signable 3-triangulation of order n yields an STS($3n$) using the following algorithm (Theorem 2.1 in [3].) This is a Bose type method.

Algorithm 1.1 *Yields an STS from a 3-triangulation*

Input: *A signable 3-triangulation of order* n Υ *with signing* σ.

Output: *A Steiner triple system* S *of order* $3n$.

Method:
1. $X \leftarrow \{(a,i)|a \in \{0,\ldots,n-1\} \textit{ and } i \in \{0,1,2\}\}$
2. $S \leftarrow \{\{(a,0),(a,1),(a,2)\}|a \in \{0,\ldots,n-1\}\}$
3. **for** *each* (a,b) *in* K_n **do**
4. $\quad S \leftarrow S \bigcup \{\{(a,j),(\Upsilon(a,b),(j\ \sigma(a,b)\ 1) \bmod 3),(b,j)\}|j=0,1,2\}$
5. *return* S

The Steiner triple system *induced* by a signable 3-triangulation Υ and a signing σ, denoted S_Υ, is the STS produced by this algorithm.

Any 3-triangulation Υ of order n can be associated with an algebra $(\{0,\ldots,n-1\},\circ)$ where $\circ$ is the operation:

$$i \circ j = \begin{cases} \Upsilon(i,j) & \text{if } i \neq j \\ i & \text{if } i = j \end{cases} \tag{1}$$

The operation $\circ$ is binary, closed, commutative, idempotent, and for each pair of distinct elements $a, b \in \{0,\ldots,n-1\}$ the equations

$$a \circ x = b \tag{2}$$
$$b \circ y = a \tag{3}$$

with unknowns x and y satisfy exactly one of the conditions:

1. There are exactly two solutions for x and none for y.
2. There are exactly two solutions for y and none for x.
3. There is exactly one solution for x and one for y.

We adopt this equivalence as an alternative definition of 3-triangulation. We often represent a 3-triangulation by the multiplication table of its algebra. Graphically we represent 3-triangulations by drawings like the one given in Figure 1. The bases in the triangles represent the edges in K_n, and the vertices opposed to the bases represent the color assigned by the 3-triangulation. This drawing contains all the structure of the 3-triangulation, in particular the triangles and adjacencies between triangles are isomorphic to the 3-triangulation's Bose cycle graph. We use signs near the bases to represent a signing of the 3-triangulation.

2 Conditions on 3-triangulations to generate anti-Pasch Steiner triple systems

In this section we present the necessary and sufficient conditions for a 3-triangulation to generate an anti-Pasch Steiner triple system.

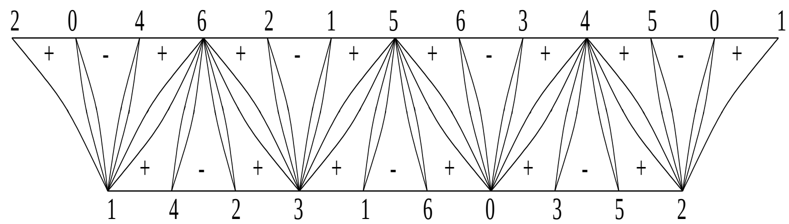

Figure 1: A 3-triangulation of order 7

Proposition 2.1 *Let $\Upsilon = (\{0,\ldots,v-1\},\circ)$ be a signable 3-triangulation of order v and let σ be one of its signings. The Steiner triple system induced by Υ and σ is anti-Pasch if and only if there do not exist four elements i,j,k,l in C satisfying one of the following conditions:*

AP-1: $i \neq k, k\circ i \neq j, j\circ(k\circ i) \neq k, j \neq (k\circ(j\circ(k\circ i)))$, $i = j\circ(k\circ(j\circ(k\circ i)))$, *and* $\sigma(k\circ(j\circ(k\circ i)),j) = \sigma(j,k\circ i) \neq \sigma(k,i) = \sigma(k,j\circ(k\circ i))$

AP-2: $i\circ j \neq j\circ k, i \neq k, i \neq j, j \neq k$, $(i\circ j)\circ(j\circ k) = i\circ k$, *and* $\sigma(i,j) = \sigma(j,k) = \sigma(i\circ j, j\circ k) \neq \sigma(i,k)$

AP-3: $i \neq j, j \neq k, k \neq l, l \neq i$, $i\circ j = k\circ l$, $i\circ l = j\circ k$, $\sigma(i,j) = \sigma(k,l)$, *and* $\sigma(i,l) = \sigma(k,j)$

AP-4: $i \neq j, j \neq (i\circ(i\circ j)), i \neq i\circ j$, $i = j\circ(i\circ(i\circ j))$, *and* $\sigma(j,i\circ(i\circ j)) = \sigma(i,i\circ j) \neq \sigma(i,j)$

AP-5: $i \neq j, i \neq i\circ j, i \neq (i\circ(i\circ j))$, $j = i\circ(i\circ(i\circ j))$, *and* $\sigma(i,j) = \sigma(i,i\circ j) = \sigma(i,i\circ(i\circ j))$

Proof: If Υ contains elements satisfying one of the conditions given, then the Steiner triple system induced by Υ and σ contains one of the Pasch configurations depicted in Figure 2. For example, in AP-1, the conditions $i \neq k, k\circ i \neq j, j\circ(k\circ i) \neq k$, and $j \neq (k\circ(j\circ(k\circ i)))$ ensure that the blocks in Figure 2(AP-1) are real triples (no edge degenerates to a vertex) and the conditions $i = j\circ(k\circ(j\circ(k\circ i)))$, and $\sigma(k\circ(j\circ(k\circ i)),j) = \sigma(j,k\circ i) \neq \sigma(k,i) = \sigma(k,j\circ(k\circ i))$ give the structure of the Pasch configuration.

The proof of the completeness of these conditions requires a routine but lengthy argument that we omit. ∎

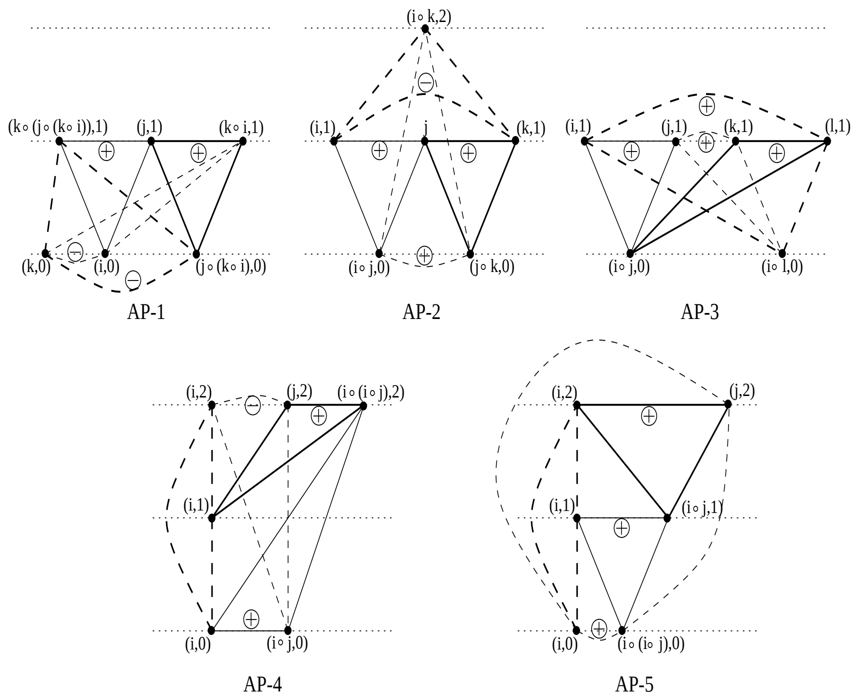

Figure 2: Pasch configurations in 3-triangulations

3 Non-sum-one-difference-zero sequences

Let $n > 1$ be an odd integer. A *non-sum-one-difference-zero sequence* of order n (NS1D0(n) for short) is a sequence $a_0, a_1, \ldots a_{(n-1)/2}$ of numbers in Z_n (with arithmetic mod n) such that:

NS1D0-1: $a_0 = 0$ and $a_{(n-1)/2} = 1$,

NS1D0-2: $\frac{1}{2}$ does not belong to the sequence,

NS1D0-3: for each $i, 1 < i < n$, $i \neq \frac{1}{2}$, only one of the numbers i or $1 - i$ belongs to the sequence, and

NS1D0-4: for each $j = 1, 2, 3, \ldots, n-1$, only one of the numbers j or $-j$ can be expressed as $a_k - a_{k-1}$ for some $k \in \{1, \ldots, (n-1)/2\}$.

The following are examples of NS1D0 sequences of orders 7, 9, 11, 13, and 15, respectively:

$$0, 5, 2, 1$$

$$0, 3, 4, 8, 1$$

$$0, 7, 9, 10, 4, 1$$

$$0, 2, 3, 10, 6, 9, 1$$

$$0, 13, 2, 11, 4, 7, 6, 1$$

As we soon see, NS1D0 sequences can be used to produce 3-triangulations, Steiner triple systems, and anti-Pasch Steiner triple systems. We expect that they can be applied to the generation of other combinatorial designs.

If $a_0, \ldots, a_{(n-1)/2}$ is a NS1D0 sequence, its *inductor* is a sequence x_1, x_2, $\ldots$, x_{n-1} such that for $1 \leq i \leq (n-1)/2$, $x_{a_i - a_{i-1}} = a_i$ and $x_{a_{i-1} - a_i} = a_{i-1}$. It follows from condition NS1D0-4 that the elements in this sequence are well defined. The inductors for the NS1D0 sequences in our examples are:

$$2, 0, 5, 2, 5, 1$$

$$4, 1, 3, 8, 4, 0, 8, 3$$

$$10, 9, 4, 0, 4, 10, 7, 1, 7, 9$$

$$3, 2, 9, 10, 1, 3, 10, 9, 6, 6, 0, 2$$

$$7, 0, 7, 2, 6, 2, 11, 4, 11, 1, 13, 4, 13, 6$$

Lemma 3.1 *Let a_0, a_1, $\ldots$, $a_{(n-1)/2}$ be a NS1D0 sequence and let x_1, x_2, $\ldots$, x_{n-1} be its inductor. Each number in the sequence different from 0 and 1 appears twice in the inductor, and* 0 *and* 1 *appear once each.*

Proof: For each $i = 1, \ldots, (n-1)/2 - 1$ only $x_{a_i - a_{i-1}}$ and $x_{a_i - a_{i+1}}$ are equal to a_i. Only $x_{a_0 - a_1}$ is equal to zero and only $x_{a_{(n-1)/2} - a_{(n-1)/2-1}}$ is equal to one. ∎

Proposition 3.2 *Each NS1D0 sequence of order n yields a 3-triangulation of order n.*

Proof: Let $a_0, a_1, \ldots, a_{(n-1)/2}$ be a NS1D0 sequence of order n and let $x_1, x_2, \ldots, x_{n-1}$ be its inductor. We claim that $(\{0, \ldots, n-1\}, \circ)$ is a 3-triangulation, where $\circ$ is the idempotent and binary operation

$$i \circ j = \begin{cases} i - x_{i-j} + \frac{1}{2} & \text{if } i \neq j. \\ i & \text{if } i = j. \end{cases}$$

The equation $i \circ \lambda = i$ with unknown λ has only i as a solution, since if there exists another solution λ_0 different from i then $i \circ \lambda_0 = i - x_{i-\lambda_0} + \frac{1}{2} = i$. Thus $x_{i-\lambda_0} = \frac{1}{2}$. But this contradicts condition NS1D0-2 because $x_{i-\lambda_0}$ belongs to $\{a_0, a_1, \ldots, a_{(n-1)/2}\}$.

The operation $\circ$ is commutative, as follows. Let i, j be two different elements in $\{0, \ldots, n-1\}$. From condition NS1D0-4 there exists a unique number k such that $a_k - a_{k-1}$ is equal to $j - i$ or $i - j$; without loss of generality we can assume that $a_k - a_{k-1} = j - i$. Then $x_{j-i} = a_k$ and $x_{i-j} = a_{k-1}$, so

$$x_{j-i} - x_{i-j} = a_k - a_{k-1} = j - i,$$
$$i - x_{i-j} = j - x_{j-i},$$

and thus

$$i \circ j = i - x_{i-j} + \tfrac{1}{2} = j - x_{j-i} + \tfrac{1}{2} = j \circ i.$$

Now consider the system of equations

$$i \circ \alpha = j \tag{4}$$
$$j \circ \beta = i \tag{5}$$

with unknowns α and β. They can be rewritten as

$$i - x_{i-\alpha} + \tfrac{1}{2} = j$$
$$j - x_{j-\beta} + \tfrac{1}{2} = i$$

or equivalently as

$$x_{i-\alpha} = (i-j) + \tfrac{1}{2}$$
$$x_{j-\beta} = (j-i) + \tfrac{1}{2}$$

Suppose first that $(i-j) + \frac{1}{2} \notin \{0, 1\}$. Then by condition NS1D0-3, only one of $(i-j) + \frac{1}{2}$ or $1 - ((i-j) + \frac{1}{2}) = (j-i) + \frac{1}{2}$ belongs to

$A = \{a_0, \ldots, a_{(n-1)/2}\}$. If $(i-j)+\frac{1}{2}$ belongs to A then, by Lemma 3.1, it appears twice in $\{x_1, x_2, \ldots, x_{n-1}\}$, and $(j-i)+\frac{1}{2}$ does not appear, so (4) has two solutions and (5) does not have any. Similarly when $(j-i)+\frac{1}{2}$ belongs to A, (5) has two solutions and (4) does not have any.

Finally, if $(i-j)+\frac{1}{2}=0$ then $(j-i)+\frac{1}{2}=1$. It follows from Lemma 3.1 that each of these elements appears once in $x_1, \ldots, x_{n-1}$, and thus each of (4) and (5) has exactly one solution. ∎

The 3-triangulation of order n produced from a NS1D0 sequence $a_0, \ldots, a_{(n-1)/2}$ is the *3-triangulation induced by the sequence.* The 3-triangulation itself is a *NS1D0 3-triangulation* with NS1D0 sequence $a_0, \ldots, a_{(n-1)/2}$.

4 NS1D0 sequences and Steiner triple systems

Here we study the signability of NS1D0 3-triangulations. Lemma 5.3 in [3] establishes that a 3-triangulation Υ is signable if and only if each cycle in its Bose graph B_Υ has an even number of negative adjacencies. So we require a characterization of the NS1D0 sequences that yield 3-triangulations with this property.

The *sign-inductor* of a NS1D0 sequence $a_0, \ldots, a_{(n-1)/2}$ is a sequence $s_1, s_2, \ldots, s_{n-1}$ such that for $1 \leq i \leq (n-1)/2$, $s_{a_i - a_{i-1}} = s_{a_{i-1}-a_i} = sign((-1)^{i-1})$.

Proposition 4.1 *Let $n > 1$ be an odd integer, and let Υ be a NS1D0 3-triangulation of order n with NS1D0 sequence $a_0, \ldots, a_{(n-1)/2}$. Υ is signable if and only if $n \equiv 3 \pmod 4$, in which case $\sigma(a,b) = s_{a-b}$ is a signing of Υ.*

Proof: Figure 3 depicts the 3-triangulation Υ and the signing σ. The number of odd adjacencies is $n((n-1)/2-1) = n(n-3)/2$. This is an even number of the form $2p$ for some positive integer p if and only if $n(n-3)/2 = 2p$ or $n(n-3) = 4p$. Since n is odd, $n-3$ must be divisible by 4, and thus $n \equiv 3 \pmod 4$. ∎

The Steiner triple system arising from the 3-triangulation and the signing in Proposition 4.1 is the *NS1D0 Steiner triple system induced by* $a_0, \ldots, a_{(n-1)/2}$ and it is denoted by $T_{a_0,\ldots,a_{(n-1)/2}}$. If $T_{a_0,\ldots,a_{(n-1)/2}}$ is anti-Pasch, then $a_0, \ldots, a_{(n-1)/2}$ is an *anti-Pasch NS1D0 sequence.*

Proposition 4.1 restricts the order n of the 3-triangulations that can be obtained from NS1D0 sequences to numbers congruent to 3 mod 4. Thus only for numbers $v = 3n \equiv 9 \bmod 12$ can it be possible to find a STS(v) induced by a NS1D0 sequence.

Table 1 gives some examples of anti-Pasch NS1D0 sequences.

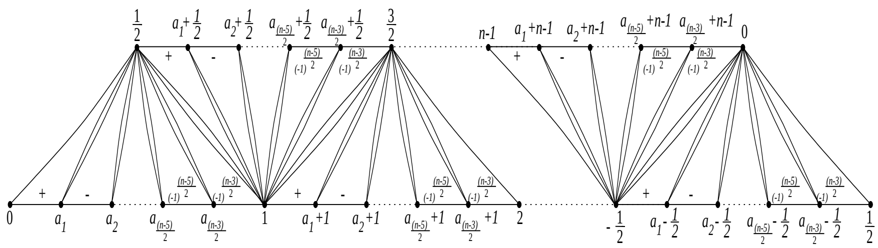

Figure 3: Bose cycle graph for a NS1D0 3-triangulation of order n

Order	NS1D0 Sequence
7	0 5 2 1
7	0 6 3 1
11	0 4 7 2 3 1
11	0 7 9 10 4 1
11	0 8 2 3 5 1
11	0 9 5 8 2 1
11	0 9 10 5 8 1
11	0 10 4 7 3 1
15	0 14 12 3 10 7 11 1
19	0 2 12 4 3 6 13 9 15 1
23	0 2 3 7 19 16 10 20 13 18 9 1
27	0 2 3 6 10 19 4 17 7 13 20 12 23 1
31	0 2 3 6 10 4 12 7 23 13 24 11 18 27 15 1
35	0 2 3 6 10 4 9 16 7 15 25 12 23 8 22 5 17 1
39	0 2 3 6 10 4 9 16 5 22 12 21 13 25 11 32 17 33 14 1
43	0 2 3 6 10 4 9 16 5 13 30 21 33 20 36 18 32 17 37 15 25 1

Table 1: Examples of anti-Pasch NS1D0 sequences

5 NS1D0 sequences and anti-Pasch Steiner triple systems

We collect some previous definitions and results. If $n \equiv 9 \pmod{12}$ is a positive integer, and $a_0, a_1, \ldots, a_{(n-1)/2}$ is a NS1D0 sequence with inductor $x_1, \ldots, x_{n-1}$ and sign-inductor $s_1, \ldots, s_{n-1}$, then $T_{a_0,\ldots,a_{(n-1)/2}}$ is the STS induced by the 3-triangulation $(0, \ldots, n-1, \circ)$ and the signing σ, where $\circ$ is the operation:

$$i \circ j = \begin{cases} i - x_{i-j} + \frac{1}{2} & \text{if } i \neq j \\ i & \text{if } i = j \end{cases} \tag{6}$$

and σ satisfies:

$$\sigma(i,j) = s_{i-j} \tag{7}$$

Proposition 5.1 *$T_{a_0,\ldots,a_{(n-1)/2}}$ is an anti-Pasch STS if and only if there do not exist two numbers $a, b \in \{0, \ldots, n-1\}$ such that one of these conditions is satisfied, with a and b chosen in such a way that the indices into x and s are all different from zero:*

AP-1':

$$-\tfrac{1}{2} + a - b + x(-\tfrac{1}{2} + b + x(-\tfrac{1}{2} - b + x(-\tfrac{1}{2} + b + x(-a)))) = 0$$

$$s(\tfrac{1}{2} - b - x(-\tfrac{1}{2} - b + x(-\tfrac{1}{2} + b + x(-a)))) = s(\tfrac{1}{2} - b - x(-a)) \neq$$
$$s(-a) = s(\tfrac{1}{2} + b - x(-\tfrac{1}{2} + b + x(-a)))$$

AP-2':

$$\tfrac{1}{2} + x(a) - x(a-b) - x(a - b - x(a-b) + x(b)) = 0$$
$$s(a-b) = s(b) = s(a - b - x(a-b) + x(b)) \neq s(a)$$

AP-3':

$$x(a) - x(b) - x(b + x(a) - x(b)) + x(a - x(a) + x(b)) = 0$$

$$s(a) = s(b)$$

$$s(b + x(a) - x(b)) = s(a - x(a) + x(b))$$

AP-4':

$$-\tfrac{1}{2} + a + x(-\tfrac{1}{2} - a + x(-\tfrac{1}{2} + x(a))) = 0$$
$$s(-\tfrac{1}{2} - a + x(-\tfrac{1}{2} + x(a))) = s(-\tfrac{1}{2} + x(a)) \neq s(a)$$

AP-5':

$$-\tfrac{1}{2} - a + x(-\tfrac{1}{2} + x(-\tfrac{1}{2} + x(a))) = 0$$
$$s(a) = s(-\tfrac{1}{2} + x(a)) = s(-\tfrac{1}{2} + x(-\tfrac{1}{2} + x(a)))$$

Here we have changed the notation x_i and s_i to $x(i)$ and $s(i)$, respectively.

Proof: Substitute (6) and (7) in the conditions of Proposition 2.1. Then make the following change of variables:

For AP-1: $a = i - k$, and $b = j - k$

For AP-2: $a = i - k$, and $b = j - k$

For AP-3: $a = i - j$, and $b = k - l$

For AP-4: $a = i - j$

For AP-5: $a = i - j$ ∎

Proposition 5.1 gives a precise specification of the anti-Pasch conditions on NS1D0 sequences. From the algorithmic point of view, it ensures that to decide if a NS1D0 sequence is not anti-Pasch, it suffices to employ two variables, a and b in the proof, both of them in the range from 0 to n. This decision takes $O(n^2)$ time, which improves upon the $O(n^4)$ time needed to check if a 3-triangulation yields an anti-Pasch STS using the implicit method in Proposition 2.1.

6 NS1D0 sequences and the n-queens problem

Finding NS1D0 sequences is strongly connected to the n-queens problem. In fact, any known method or heuristic to find a solution to the n-queens problem yields an analogous method or heuristic to find NS1D0 sequences.

To explain this equivalence we interpret an NS1D0 sequence of order n as an arrangement of "queens" on a "chess-board" of size $n \times n$. The rows and columns of the chess-board are indexed from 0 to $(n-1)$ and each diagonal in the $\searrow$ direction containing one cell with coordinates (i, j) is labeled with the number $(j - i)$. Thus, given a sequence $a_0, a_1, \ldots, a_{(n-1)/2}$ (not necessarily a NS1D0 sequence) of pairwise different numbers in $\{0, \ldots, n\}$, its *queen arrangement* is the set $\mathcal{Q} = \{(a_i, a_{i+1}) | i = 0, \ldots, (n-1)/2 - 1\}$ (the set of queens). In this arrangement the row i is *occupied* if and only if there exists $j \in \{0, \ldots, n-1\}$ such that $(i, j) \in \mathcal{Q}$. When this occurs, (i, j)

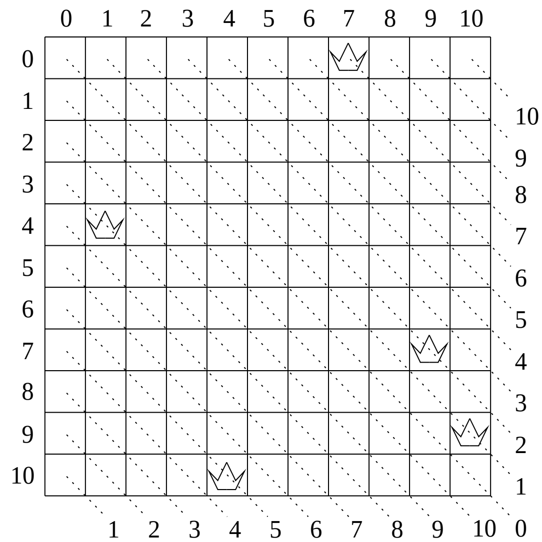

Figure 4: Queen arrangements for the NS1D0(11) 0,7,9,10,4,1

occupies row i. A row that is not occupied by any $i = 0, \ldots, n-1$ is *empty* or *free*. Occupancy and emptiness for columns and diagonals are defined in a similar way. Figure 4 gives an example of n-queen arrangement.

The characterization of queen arrangements corresponding to NS1D0 sequences is immediate:

Proposition 6.1 *Let $n > 1$ be an odd integer and let $a_0, a_1, \ldots, a_{(n-1)/2}$ be a sequence of numbers in $\{0, \ldots, n-1\}$. This is a NS1D0 sequence if and only if its n-queen arrangement satisfies the following conditions:*

NS1D0-1': Row 1 and column 0 are free, but row 0 and column 1 are occupied by exactly one queen.

NS1D0-2': Row $\frac{1}{2}$ and column $\frac{1}{2}$ are free.

NS1D0-3': For each $j \in 2, \ldots, n-1$, exactly one queen occupies some of the columns j and $1-j$. The same is true for rows. Column j is occupied if and only if row j is occupied.

NS1D0-4': For each $j \in 1, \ldots, n-1$, exactly one queen occupies diagonal j or $-j$.

A n-queen arrangement satisfying NS1D0-1' to NS1D0-4' is a *NS1D0 n-queen arrangement*.

Conditions NS1D0-1' to NS1D0-4' are similar to the check rules in the n-queens problem. In a NS1D0 n-queen arrangement the dominance of each queen (that does not occupy columns and rows 0,1) spans two columns, two rows and four $\searrow$ diagonals, but two queens in the same $\swarrow$ diagonal do not dominate each other.

For a NS1D0 sequence of order n its n-queen arrangement is a *NS1D0 n-queen arrangement*; for an anti-Pasch sequence it is an *anti-Pasch n-queen arrangement.* Our interest is in the following problems:

Problem 6.1 (NS1D0 n-queens problem.) *Given an odd integer $n > 1$ find all NS1D0 n-queen arrangements.*

Problem 6.2 (Anti-Pasch n-queen problem) *Given an odd integer $n > 1$ with $n \equiv 3 \pmod 4$, find all anti-Pasch n-queen arrangements.*

We describe an algorithmic solution to both problems. Our approach is based in the classic iterative method to solve the n-queens problem. Equivalent modifications could be done to any other known method.

In this algorithm we assume the existence of a data structure to record the occupancy of rows, columns and diagonals. All rows, columns and diagonals are free when execution starts. The method looks for all possible queen arrangements that satisfy NS1D0-1' through NS1D0-4', so those output are in fact all possible NS1D0 sequences of order n.

Algorithm 6.1 *Solve the NS1D0 n-queens problem.*

Input: An odd integer $n > 1, n \equiv 3 \bmod 4$.

Output: All NS1D0 sequences of order n.

Method:

Mark rows 1 and $\frac{1}{2}$, columns 0 and $\frac{1}{2}$ and diagonal 0 as occupied.
a[0]$\leftarrow$ 0
$placed\_queens \leftarrow 0$
$column\_occupied\_in\_row[0] \leftarrow -1$
while $placed\_queens \geq 0$ **do**
    $row \leftarrow a[placed\_queens]$
    $column\_occupied\_in\_row[row] \leftarrow column\_occupied\_in\_row[row] + 1$
    $column \leftarrow column\_occupied\_in\_row[row]$
    if $column \geq n$ **then**
        $column\_occupied\_in\_row[row] \leftarrow 0$
        $placed\_queens \leftarrow placed\_queens - 1$
        if $placed\_queens < 0$ **then** continue
        $row \leftarrow a[placed\_queens]$

14. $\quad column \leftarrow column\_occupied\_in\_row[row]$
15. $\quad$ **if** $row \neq 0$ **then** mark row $(1 - row)$ as free
16. $\quad$ **if** $column \neq 1$ **then** mark column $(1 - column)$ as free
17. $\quad$ Mark row row, column $column$ and diagonals
 $\quad\quad (row - column)$ and $(column - row)$ as free.
18. $\quad$ continue
19. **if** $((column = 1)$ AND $(placed\_queens < (n-1)/2 - 1))$ OR
 (row row is occupied) OR (column $column$ is occupied) OR
 (one of the diagonals $(row - column)$ or $(row - column)$ is
 occupied) OR $(column = 1 - row)$ **then** continue
20. $placed\_queens \leftarrow placed\_queens + 1$
21. $a[placed\_queens] \leftarrow column$
22. $column\_occupied\_by\_row[column] \leftarrow -1$
23. **if** $row \neq 0$ **then** mark row $(1 - row)$ as occupied.
24. Mark row row, columns $column$ and $(1 - column)$,
 and diagonals $(row - column)$ and $(column - row)$ as occupied.
25. **if** $placed\_queens = (n-1)/2$ **then**
26. $\quad$ display $a[0], \ldots, a[(n-1)/2]$

An algorithmic solution to the anti-Pasch n-queens problem can be obtained by an easy modification to Algorithm 6.1 if we check at each iteration that the equations in Proposition 5.1 are satisfied. Since we are gradually building the sequence, some parts of the inductors are not available and some equations cannot be evaluated, but if some equation which can be evaluated is not satisfied then we can backtrack immediately. In practice this is an efficient pruning method. This modification can be introduced by adding the following line to Algorithm 6.1:

19.5 **if** some equation in Proposition 5.1 can be evaluated and is not satisfied **then** continue

7 Conclusions

The construction of anti-Pasch Steiner triple systems by direct application of Bose's method is complicated by the apparent difficulty of determining when a particular quasigroup leads to an anti-Pasch system. NS1D0 sequences, in contrast, arise from algebraic structures for which the anti-Pasch conditions can be expressed precisely. The close relationship between the NS1D0 n-queens problem and the n-queens problem together with the anti-Pasch conditions of Proposition 5.1 opens a new way to generate anti-Pasch STSs. Indeed, NS1D0 sequences are so well structured that appears probable that explicit constructions of anti-Pasch NS1D0 sequences can be developed.

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